Tutorial 5 (24 Feb)

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Cheat Sheet for physical quantities

2-dimensional version

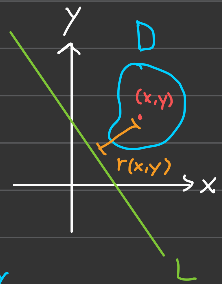
Given the density function $\delta : D \longrightarrow \mathbb{R}$ over a region $D \subseteq \mathbb{R}^2$,

- Mass: $M := \iint_D \delta(x,y)\, dA$
- First moments:
  - $M_y := \iint_D x\,\delta(x,y)\, dA$
  - $M_x := \iint_D y\,\delta(x,y)\, dA$
- Center of mass: $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) \in \mathbb{R}^2$

- Moment of inertia
  - about the x-axis: $I_x := \iint_D y^2 \delta(x,y)\, dA$
  - about the y-axis: $I_y := \iint_D x^2 \delta(x,y)\, dA$
  - about a line L: $I_L := \iint_D r^2(x,y)\, \delta(x,y)\, dA$, where

$$\begin{array}{ccc} r : D & \longrightarrow & \mathbb{R} \\ \cup\!\!\!\!\!\!\;\text{\scriptsize|} & & \cup\!\!\!\!\!\!\;\text{\scriptsize|} \\ (x,y) & \longmapsto & r(x,y) \end{array}$$

$r(x,y)$: distance between (x,y) and L



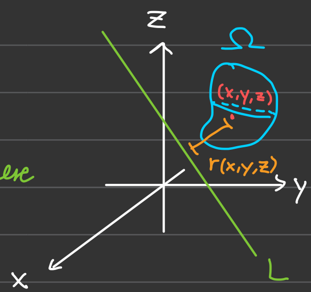
  - about the origin: $I_o := \iint_D (x^2+y^2)\,\delta(x,y)\, dA = I_x + I_y$.

3-dimensional version

Given the density function $\delta : \Omega \longrightarrow \mathbb{R}$ over a solid $\Omega \subseteq \mathbb{R}^3$,

- Mass: $M := \iiint_\Omega \delta(x,y,z)\, dV$
- First moments:
  - $M_{yz} := \iiint_\Omega x\, \delta(x,y,z)\, dV$
  - $M_{xz} := \iiint_\Omega y\, \delta(x,y,z)\, dV$
  - $M_{xy} := \iiint_\Omega z\, \delta(x,y,z)\, dV$
- Center of mass: $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right) \in \mathbb{R}^3$

- Moment of inertia
  - about the x-axis: $I_x := \iiint_\Omega (y^2+z^2)\, \delta(x,y,z)\, dV$
  - about the y-axis: $I_y := \iiint_\Omega (x^2+z^2)\, \delta(x,y,z)\, dV$
  - about the z-axis: $I_z := \iiint_\Omega (x^2+y^2)\, \delta(x,y,z)\, dV$
  - about a line L : $I_L := \iiint_\Omega r^2(x,y,z)\, \delta(x,y,z)\, dV$, where



$$\begin{array}{ccc} r : \Omega & \longrightarrow & \mathbb{R} \\ \cup\!\!\!\!\!\;\; & & \cup\!\!\!\!\!\;\; \\ (x,y,z) & \longmapsto & r(x,y,z) \end{array}$$

$r(x,y,z)$: distance between (x,y,z) and L

Fubini's Theorem for triple integrals in cylindrical coordinates

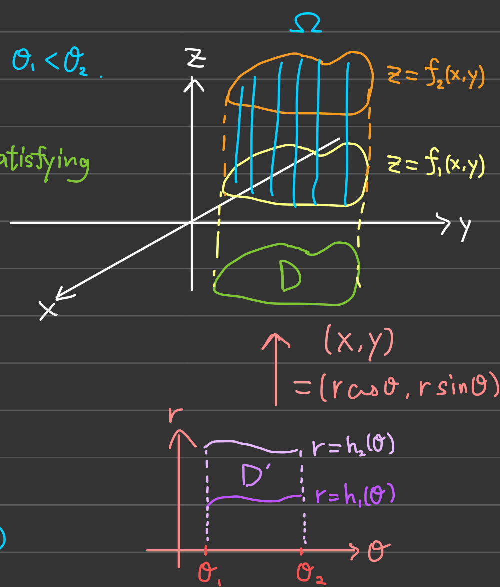
Thm (Fubini's Theorem for continuous functions in cylindrical coordinates)

Let $g : \Omega \to \mathbb{R}$ be a continuous function over a solid Ω, where

- $\Omega := \{(x,y,z) \in \mathbb{R}^3 \mid (x,y) \in D \,;\, f_1(x,y) \leq z \leq f_2(x,y)\} \subseteq \mathbb{R}^3$, where
  - $D = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2 \mid \theta_1 \leq \theta \leq \theta_2 \,;\, h_1(\theta) \leq r \leq h_2(\theta)\}$, where
    - $\theta_1, \theta_2 \in [0, 2\pi)$ are constants satisfying $\theta_1 < \theta_2$.
    - $h_1, h_2 : [\theta_1, \theta_2] \to \mathbb{R}$ are continuous satisfying

      $0 \leq h_1(\theta) \leq h_2(\theta)$ for any $\theta \in [\theta_1, \theta_2]$.
  - $f_1, f_2 : D \to \mathbb{R}$ are continuous

    with $f_1(x,y) \leq f_2(x,y)$, $\forall (x,y) \in D$



then
$$\iiint_\Omega g \, dV = \iint_D \left(\int_{f_1(x,y)}^{f_2(x,y)} g(x,y,z)\, dz \right) dA(x,y)$$
$$= \int_{\theta_1}^{\theta_2} \int_{h_1(\theta)}^{h_2(\theta)} \int_{f_1(r\cos\theta, r\sin\theta)}^{f_2(r\cos\theta, r\sin\theta)} g(r\cos\theta, r\sin\theta, z)\, r \, dz\, dr\, d\theta$$

Fubini's Theorem for triple integrals in spherical coordinates

Thm (Fubini's Theorem for continuous functions in spherical coordinates)

Let $f:\Omega \to \mathbb{R}$ be a continuous function over a solid Ω, where

- $\Omega := \{\Phi(\rho,\varphi,\theta)=(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\in\mathbb{R}^3 \mid (\rho,\varphi,\theta)\in\Omega_1\}\subseteq\mathbb{R}^3$, where

  $\Omega_1\subseteq[0,+\infty)\times[0,\pi]\times[0,2\pi)\subseteq\mathbb{R}^3$ is a solid in (ρ,φ,θ)-space

such that
$$\begin{array}{ccc} [0,+\infty)\times[0,\pi]\times[0,2\pi) & \xrightarrow{\Phi(\rho,\varphi,\theta)} & \mathbb{R}^3 \\ \cup\!\!\mid & & \cup\!\!\mid \\ \Omega_1 & \xrightarrow{\cong} & \Omega \end{array}$$

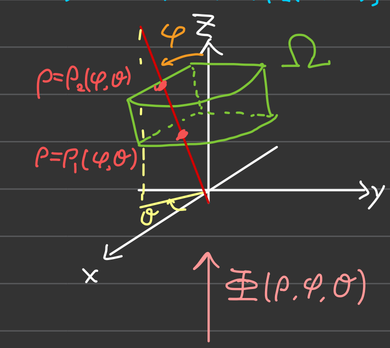
then $\displaystyle\iiint_{\Omega} f(x,y,z)\,dV(x,y,z) = \iiint_{\Omega_1} f(\Phi(\rho,\varphi,\theta))\,\rho^2\sin\varphi\,dV(\rho,\varphi,\theta)$

In particular, if $\Omega_1=\{(\rho,\varphi,\theta)\mid \underbrace{\theta_1\leq\theta\leq\theta_2}_{},\ \varphi_1\leq\varphi\leq\varphi_2,\ \rho_1(\theta,\varphi)\leq\rho\leq\rho_2(\theta,\varphi)\}$

(If $\theta_1=0$; $\theta_2=2\pi$, replaced by $0\leq\theta<2\pi$)

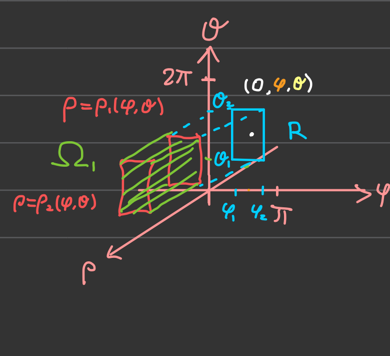
- $\theta_1,\theta_2\in[0,2\pi]$ are constants satisfying $\theta_1<\theta_2$.
- $\varphi_1,\varphi_2\in[0,\pi]$ are constants satisfying $\varphi_1<\varphi_2$.
- $\rho_1,\rho_2: R:=[\varphi_1,\varphi_2]\times[\theta_1,\theta_2]\to\mathbb{R}$ are continuous

with $0\leq\rho_1(\varphi,\theta)\leq\rho_2(\varphi,\theta)$, $\forall(\varphi,\theta)\in R$.



then $\displaystyle\iiint_{\Omega} f(x,y,z)\,dV(x,y,z)$

$$=\int_{\theta_1}^{\theta_2}\int_{\varphi_1}^{\varphi_2}\int_{\rho_1(\varphi,\theta)}^{\rho_2(\varphi,\theta)} f(\Phi(\rho,\varphi,\theta))\,\rho^2\sin\varphi\,d\rho\,d\varphi\,d\theta$$



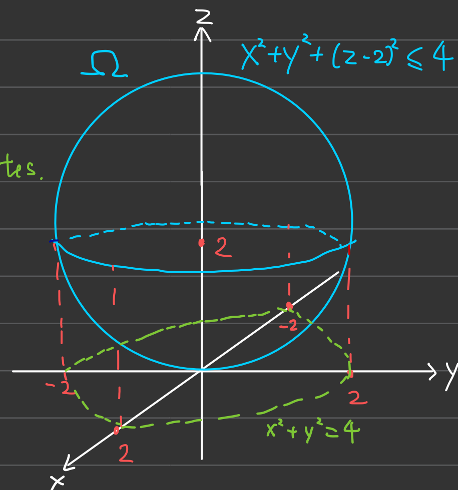
Ex Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{2-\sqrt{4-x^2-y^2}}^{2+\sqrt{4-x^2-y^2}} (x^2+y^2+z^2)^{\frac{3}{2}}\,dz\,dy\,dx$.

Sol Idea: Understand the integral better and apply a change of coordinates.

Step 1 Describe the domain of integration Ω in $\mathbb{R}^3$.

$$\Omega = \{(x,y,z)\in\mathbb{R}^3 \mid -2\le x\le 2,\ -\sqrt{4-x^2}\le y\le\sqrt{4-x^2},\ 2-\sqrt{4-x^2-y^2}\le z\le 2+\sqrt{4-x^2-y^2}\}$$

Step 2 Sketch Ω in $\mathbb{R}^3$.



Step 3 Describe Ω in terms of spherical coordinates.

$$\begin{cases} x = \rho\sin\varphi\cos\theta \\ y = \rho\sin\varphi\sin\theta \\ z = \rho\cos\varphi \end{cases}$$

Note that $x^2+y^2+(z-2)^2\le 4 \iff x^2+y^2+z^2\le 4z$

In spherical coordinates: $\rho^2\le 4\rho\cos\varphi \iff \rho\le 4\cos\varphi$.

Also, $z\ge 0 \iff \rho\cos\varphi\ge 0 \iff 0\le\varphi\le\frac{\pi}{2}$.

$\therefore\ \Omega = \{(\rho\sin\varphi\cos\theta,\ \rho\sin\varphi\sin\theta,\ \rho\cos\varphi)\in\mathbb{R}^3 \mid (\rho,\varphi,\theta)\in\Omega_1\}$, where

$$\Omega_1 = \{(\rho,\varphi,\theta)\in[0,+\infty)\times[0,\pi]\times[0,2\pi) \mid 0\le\theta<2\pi,\ 0\le\varphi\le\frac{\pi}{2},\ 0\le\rho\le 4\cos\varphi\}$$

Step 4 Evaluate the integral using spherical coordinates.

$$\text{Integral} = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{4\cos\varphi} (\rho^2)^{\frac{3}{2}} \cdot (\rho^2 \sin\varphi)\, d\rho\, d\varphi\, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{2}}\int_0^{4\cos\varphi} \rho^5 \sin\varphi\, d\rho\, d\varphi\right)$$

$$= 2\pi \cdot \int_0^{\frac{\pi}{2}} \left[\frac{\rho^6}{6} \cdot \sin\varphi\right]_0^{4\cos\varphi} d\varphi$$

$$= 2\pi \cdot \frac{4096}{6} \int_0^{\frac{\pi}{2}} \cos^6\varphi \sin\varphi\, d\varphi$$

$$= \frac{4096}{3}\pi \left[-\frac{\cos^7\varphi}{7}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{4096}{3}\pi\left(\frac{1}{7}\right) = \frac{4096}{21}\pi$$